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Research article

Sensitivity Analysis on Significant Performance Measures of Bulk Arrival Retrial Queueing $M^X/(G_1,G_2)/1$ Model with Second Phase Optional Service and Bernoulli Vacation Schedule

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Abstract

Present paper envisages some significant performance measures of a bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model with two phase service wherein first phase service is essential and the next second phase service is optional. In this paper, the concept of Bernoulli vacation schedule has been incorporated to utilize the vacation period of server to complete some additional work in the case when next second optional phase service is not needed by arriving customers and such type of vacation is referred as working vacation. We assume with realistic provision that the server has an option to avail a vacation with probability p (q) or may continue to serve the next customer, if any with complementary probability $\bar{p}\,(\bar{q})$ just after the completion of first phase essential service i.e. before the commencement of second phase optional service. Specifically, we have explored many performance measures including system length, orbit length, waiting time etc. In addition to this, we have also examined some numerical illustrations and its sensitivity analysis by way of two dimensional graphs for three different types of service time distributions such as Deterministic, Exponential and Gamma distributions; which may be useful to the system designers and decision makers in the emerging fields of Science, Engineering and Technology. Finally, based on numerical experiment performed, overall we observed that the varying trends in mean orbit size are more perceptible for Gamma service time distribution as compared to Erlangian and Deterministic service time distributions.

Keywords: Bulk arrival retrial queueing model, Bernoulli vacation schedule, Exponential and Gamma distributions, Performance measures.

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1. Introduction

In the wide range of queueing situations, retrial phenomenon is generally used to many congestion situations encountered in computer networks, telecommunication distribution, production and manufacturing systems and various daily life situations. Any arriving batch enters a virtual pool of blocked customers called 'orbit' when the server is busy or in working vacation; otherwise one customer from the arriving batch gets the service immediately while the rest customers join the retrial group (orbit). Here it is worth mentioning that several noteworthy researchers [1, 2, 3…33, 34] and references therein confined their attention to examine exhaustively the queueing models with and without retrial arrivals in diverse frameworks. Artalejo [2], Kulkarni [21] and Templeton [29] have given categorical survey on retrial queueing systems. Chakravarthy and Dudin [7] studied a retrial queueing model in which two types of customers arrive according to a Markovian arrival process. Wu *et al.* [34] studied the retrial queues with general service times and non-exponential retrial time distribution. Sherman and Kharoufeh [28] analyzed an unreliable *M/M/1* retrial queue with infinite-capacity orbit and succeeded to investigate the stability conditions as well as several stochastic decomposability results. Moreover, Wang *et al.* [33] examined an *M/G/1* retrial queueing system with disasters and unreliable server and investigated the Laplace transforms both of the transient solutions and steady-state solutions for queueing and reliability measures of interest. Furthermore, recently Amandor and Artalejo [1] have focused their attention to study on an *M/G/1* retrial queue to determine the distribution of the successful and blocked events made by the primary customers and the retrial customers. Further, by the same time Boualem *et al.* [6] considered an *M/G/1* retrial queue with vacations and they investigated several stochastic comparison properties for the stationary queue length distribution.

Both idle as well as busy periods of a server in various queueing models pay vital role particularly due to service cost. As a consequence, utility of idle periods in queueing models with server vacations has great importance by the fact that the idle period of the server may be used for doing some other secondary work. A central attention in this connection has been paid by Maurya [25] to explore the expected busy periods in faster and slower arrival rates of an interdependent M/M/1:(∞; GD) queueing model with controllable arrival rates. In order to make more realistic and versatile for analyzing the real-world congestion problems, servers are allowed to take vacations for fixed or variable periods. Applications of such retrial queueing model can be found in production systems, data communication networks and call centers etc. A wide range of works done on retrial queue with server vacation model can be found in queueing literature. For evidence in this connection, we refer the noble book by Tian and Zhang [31]. The comprehensive survey on this topic has been established in Doshi [13, 14]. Later, Choudhury and Madan [10] considered a bulk arrival queueing system wherein the server delivers two phases of heterogeneous service and succeeded to investigate the queue size distribution at random epochs of the system states along with various vital performance measures. Moreover, Atencia and Moreno [5] discovered an *M/G/1* retrial queue with general retrial times with Bernoulli schedule and derived the generating function of the system size distribution and explored also the stochastic decomposition law. Thiagarajan and Srinivasan [30] considered to analyze an interdependent $M^x/M/1/\infty$ queueing model with controllable arrival rates. Choudhury and Deka [8] considered an $M^X/G/1$ queueing model with two phases of heterogeneous service under Bernoulli vacation schedule and classical retrial policy. By making use of the embedded Markov chain technique, Choudhury and Deka [8] determined the steady state distribution of the server state and the number of the customers in the retrial group. Maurya [24] examined to explore some useful results for an $M^x/E_k/1/\infty$ queueing model with bulk arrivals and Erlangian service time distribution. Recently, Choudhury and Deka [9] analyzed rigorously the steady state behavior of an $M^X/G/1$ unreliable retrial queue with Bernoulli admission mechanism.

It has been critically observed that in many realistic queueing situations, usually jobs demand the first phase “essential” service, whereas only some of them demand the next phase “optional” service. Wang [32] analyzed an *M/G/1* queue with second phase optional service and unreliable server and they achieved to establish both the transient and steady-state solutions by using a supplementary variable technique. Later, a single server queue with two phases of heterogeneous service and linear retrial policy under Bernoulli vacation schedule was analyzed by Madan and Choudhury [23]. Moreover, Choudhury and Paul [11] considered to examine a queueing model wherein the server provides two phases of heterogeneous service to each customer in succession with Bernoulli vacation schedule under different vacation policies. Furthermore, Choudhury *et al.* [12] analyzed the steady state behavior of a bulk arrival queue and Bernoulli schedule vacation under multiple vacation policy and they obtained successfully the queue size distribution of idle period process. Of late, the steady state behavior of an *M/G/1* retrial queue with an additional second phase of optional service was also examined by Choudhury and Deka [8]. Besides these significant works, some other noteworthy researchers have paid their keen interest to explore a variety of retrial queue with different versions. Among them, Dudin [15] considered a novel retrial multi-server queueing model with batch arrival of customers taking into account that the customers of a batch arrive one-by one in exponentially distributed times in contrary to the standard batch arrival when a whole batch arrives into the system at one epoch and succeeded to explore the joint distribution of the number batches and customers in the system using the tool of the multi-dimensional asymptotically quasi-Toeplitz Markov chains. Ke and Chang [19] investigated a bulk arrival retrial queue with general retrial times where the server offers two phases of heterogeneous service to all the customers under Bernoulli vacation schedules. Ke and Lin [20] examined the $M^X/G/1$ queueing system with server vacations and they investigated a comparative analysis between the approximate results with established exact results for vacation time, service time and repair time distributions by using the principle of maximum entropy. Recently, Arivudainambi and Godhandaraman [4] analyzed a batch arrival queueing system with two phases of service, feedback and *K* optional vacations under a classical retrial policy and they explored the steady state distribution of the server state and the number of customers in the orbit along with numerical illustration to examine the effects of various parameters on the system performance. Very recently, Maurya [26] studied a bulk retrial arrival queueing $M^X/(G_1,G_2)/1$ model incorporating three features:

(i) State dependent arrival rates
(ii) Working vacation and
(iii) Second phase optional service

and he succeeded to investigate probability generating functions for first phase essential service (FPES), next phase optional service (NPOS) and working vacation and also for the number of the customers in the orbit at an arbitrary epoch.

Although, a lot of work has been done in retrial queueing systems by a large number of noteworthy researchers, yet no one of previous researchers has confined his attention to find the system length, orbit length and waiting time distribution for a bulk arrival retrial queue with general retrial time and two phases of service under Bernoulli vacation schedule. To fill this gap, we have given a mathematical description in section-2; the practical justification for the model is discussed in sections 4-6 in this paper. Here, we are interested to investigate some significant performance measures of a bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model studied by Maurya [26]. Specifically, we have obtained the system length, orbit length, waiting time and also discussed for its sensitivity analysis to verify our investigated results. In addition to this, we have also examined briefly some special cases along with numerical illustrations for three different types of service time distributions such as Deterministic, Exponential and Gamma distributions; which are consistent with the existing literature. We remark here that our results in special cases agree with the results obtained by Falin [16] and Kumar and Arumuganathan [22].

2. Description and Notations of the Retrial Queueing $M^X/(G_1,G_2)/1$ Model

In the present paper, we envisage a single server retrial queueing system with first phase essential and second phase optional service. The server provides his services in two phases, where service of the first phase is essential, however, service of the next second phase is optional. We presume that all the arriving customers have to get the essential service in first phase whereas next second phase optional service (SPOS) is provided only to those customers who demand for the same. As soon as the FPES (SPOS) of the customers is completed, the server may go for vacation with probability *p* (*q*) or may continue to serve the next customer, if any with probability $\bar{p}\,(\bar{q})$. During his vacation period, the server may undertake some additional work with a different service rate and such type of vacation of server is assumed as working vacation. After the completion of FPES if the customer demands for the SPOS, then server may provide the SPOS with probability σ or becomes idle with probability $\bar{\sigma}$. We assume that the customers arrive in batches with a fixed batch size according to Poisson process with batch size distribution c_j and service times of FPES, SPOS and working vacation are distributed according to general service time distribution with mean service times $\frac{1}{\mu_1},\frac{1}{\mu_2},\frac{1}{\mu_3}$ respectively. In the retrial group, the time between the two successive attempts of each customer is considered to be exponentially distributed with rate ν. For the sake of presentation and mathematical formulation of the model, Let us consider a set of following assumptions:

X; the random variable denoting the batch size with batch size distribution as defined by

$$c_j = \Pr[c = k],\ k = 1,2,.......,d$$

and the generating function for the batch size distribution is given as follows

$$\mathrm{C(z)} = \sum_{k=1}^{\infty} z^k c_k$$

which possess its mean and variance respectively

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$$C'(1) = \text{E[X] and } C''(1) = E[X^2]$$

$N(t)$; the number of customers present in the system at time t

$A(t)$; the random variable denoting the server's state at time t; where $A(t)$ is defined as following for different states:

$$A(t) = \begin{cases} 0, & \text{if the server is in idle state} \\ 1, & \text{if the server is in FPES state} \\ 2, & \text{if the server is in SPOS state} \\ 3, & \text{if the server is on working vacation state} \end{cases}$$

Moreover, we denote the state dependent arrival rates of the customers by symbol λ_i; given as follows:

$$\lambda_i = \begin{cases} \lambda_0, & \text{when the server is in idle state} \\ \lambda_1, & \text{when the server is in FPES state} \\ \lambda_2, & \text{when the server is in SPOS state} \\ \lambda_3, & \text{when the server is on working vacation state} \end{cases}$$

In addition to these, we use following notations for cumulative distribution function (CDF), probability distribution function (PDF), Laplace-Stieltijes transformation (LST) and the remaining service time (RST) or remaining working vacation time (RVT), respectively of FPES, SPOS and working vacation.

State	**CDF**	**PDF**	**LST**	**RST/RVT**
FPES	$S_1(x)$	$s_1(x)$	$\tilde{S}_1(\theta)$	$S_1^0(x)$
SPOS	$S_2(x)$	$s_2(x)$	$\tilde{S}_2(\theta)$	$S_2^0(x)$
Working Vacation	$S_3(x)$	$s_3(x)$	$\tilde{S}_3(\theta)$	$S_3^0(x)$

The steady state probabilities are defined as following: $P_{0,n}(t)dt = \Pr\{N(t) = n, A(t) = 0\}, \quad n \geq 0$

$P_{i,n}(x,t)dt = \Pr\{N(t) = n, A(t) = i, \; x \leq S_i^0(t) \leq x + dx\},$

$n \geq 0, \; i = 1,2,3.$

The r[th] moment of FPES, SPOS and working vacation states are denoted by $E[S_1^r], E[S_2^r], \text{ and } E[S_3^r]$, where $r \geq 1$.

Thus, we have following expressions to obtain $E[S_i^r]$; $i = 1,2,3.$

$$E[S_1^r] = (-1)^r S_1'^{(r)}(0); \; E[S_2^r] = (-1)^r S_2'^{(r)}(0); \; E[S_3^r] = (-1)^r S_3'^{(r)}(0); \; r \geq 1$$

We define the probability generating functions (PGF)

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{0,n}$$

3. Preliminaries of the Retrial Queueing $M^X/(G_1,G_2)/1$ Model

In this section, the probability generating function of the number of customers in the orbit is presented in the form of following theorem 3.1 in view of Maurya [26] which we shall use here to further explore some useful performance measures of a bulk arrival retrial queueing $M^X/(G_1, G_2)/1$ model.

Theorem 3.1: The probability generating function of the number of the customers in the orbit is given by

$$P(z)=\frac{P_0(z)}{\psi}\begin{bmatrix}\psi-\lambda_0(\lambda_2\lambda_3(\tilde{S}_1(\lambda_1-\lambda_1C(z)-1)\\+\lambda_1\lambda_3\bar{p}\sigma\tilde{S}_1(\lambda_1-\lambda_1C(z)(\tilde{S}_2(\lambda_2-\lambda_2C(z)-1)\\+\lambda_1\lambda_2\tilde{S}_1(\lambda_1-\lambda_1C(z)\\\times(\tilde{S}_3(\lambda_3-\lambda_3C(z)-1)(\bar{p}\sigma q\tilde{S}_2(\lambda_2-\lambda_2C(z))+\bar{\sigma})\end{bmatrix}$$

where $P_0(z)$, $P_0(1)$, ψ and ρ are as follows

$$P_0(z)=P_0(1)$$
$$\times\exp\left[-\frac{\lambda_0}{\nu}\int_z^1\left(\frac{\begin{matrix}1-\frac{C(u)}{u}[\tilde{S}_1(\lambda_1-\lambda_1C(u))(\bar{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))\\\times(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\bar{q})+p\bar{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\bar{p}\bar{\sigma}))]\end{matrix}}{\begin{matrix}[\tilde{S}_1(\lambda_1-\lambda_1C(u))(\bar{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(u))\\\times(q\tilde{S}_3(\lambda_3-\lambda_3C(u))+\bar{q})+p\bar{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(u))\\+(p\sigma+\bar{p}\bar{\sigma}))]-u\end{matrix}}\right)du\right]$$

where

$$P_0(1)=\frac{1-\rho}{[1-\rho+\lambda_0E[X][E[S_1]+\bar{p}\sigma E[S_2]+(\bar{p}\sigma q+\bar{\sigma})E[S_3]]}$$

$$\psi=\lambda_1\lambda_2\lambda_3\left[\tilde{S}_1(\lambda_1-\lambda_1C(z))\right]\times\begin{pmatrix}\bar{p}\sigma\tilde{S}_2(\lambda_2-\lambda_2C(z))(q\tilde{S}_3(\lambda_3-\lambda_3C(z))+\bar{q})\\+p\bar{\sigma}\tilde{S}_3(\lambda_3-\lambda_3C(z))+(p\sigma+\bar{p}\bar{\sigma})\end{pmatrix}$$

$$\rho=\bar{p}\sigma E[X]\begin{bmatrix}\lambda_1E[S_1]+\lambda_2E[S_2]\\+q\lambda_3E[S_3]\end{bmatrix}+p\bar{\sigma}E[X][\lambda_1E[S_1]+\lambda_3E[S_3]]+(p\sigma+\bar{p}\bar{\sigma})\lambda_1E[S_1]E[X]$$

4. Performance Measures of the Retrial Queueing $M^X/(G_1,G_2)/1$ Model

In this section, we derive the expressions for some significant performance measures to examine the behavior of the system taken into present consideration by way of using theorem 3.1 of section-3. To serve our present purpose, some significant performance measures such as mean number of the customers in the retrial group (orbit) and mean waiting time in the orbit are stated in the following theorems 4.1-4.2

Theorem 4.1: The mean number of the customers in the orbit for the bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model is given by following expression

$$E[N]=\left[\begin{array}{l}\dfrac{\lambda_0(\rho+E[X]-1)(\psi'-\lambda_0\rho_1)}{\nu(1-\rho)\psi'}\\ +\dfrac{\psi''-\lambda_0\left(\eta_1+\dfrac{E[X^2]\rho_1}{E[X]}\right)}{2\psi'}\end{array}\right]P_0(1)$$
$$-\frac{\psi''(\psi'-\lambda_0\rho_1)}{2(\psi')^2}P_0(1)$$

where

$$\psi'=\lambda_1\lambda_2\lambda_3(\rho-1)$$
$$\psi''=\lambda_1\lambda_2\lambda_3\left((E[X])^2\eta+\frac{E[X^2]\rho}{E[X]}\right)$$

$$\eta=\bar{p}\sigma\left[\begin{array}{l}\lambda_1^2E[S_1^2]+\lambda_2^2E[S_2^2]+q\lambda_3^2E[S_3^2]\\ +2\lambda_1\lambda_2E[S_1]E[S_2]+2q\lambda_1\lambda_3E[S_1]E[S_3]\\ +2q\lambda_2\lambda_3E[S_2]E[S_3]\end{array}\right]$$
$$+p\bar{\sigma}\left[\lambda_1^2E[S_1^2]+\lambda_3^2E[S_3^2]+2\lambda_1\lambda_3E[S_1]E[S_3]\right]$$
$$+\lambda_1^2E[S_1^2](p\sigma+\bar{p}\bar{\sigma})$$

$$\rho_1=\lambda_1\lambda_2\lambda_3E[X]\left[\begin{array}{l}E[S_1]+\bar{p}\sigma E[S_2]\\ +(\bar{p}\sigma q+\bar{\sigma})E[S_3]\end{array}\right]$$

$$\eta_1=\lambda_1\lambda_2\lambda_3\left[\begin{array}{l}\lambda_1E[S_1^2]+\bar{p}\sigma\lambda_2E[S_2^2]\\ +(\bar{p}\sigma q+\bar{\sigma})\lambda_3E[S_3^2]\\ +2\lambda_1\bar{p}\sigma E[S_1]E[S_2]\\ +2\lambda_1(\bar{p}\sigma q+\bar{\sigma})E[S_1]E[S_3]\\ +2\lambda_2\bar{p}\sigma qE[S_3]E[S_2]\end{array}\right]$$

Proof: To prove theorem 4.1 in order to find $E[N]$; the mean number of the customers in the orbit, we proceed by using following property of probability generating function $P(z)$ as given in equation (4.1)

$$E[N]=\sum_{n=0}^{\infty}nP_n=\lim_{z\to 1}P'(z) \tag{4.1}$$

In the light of theorem 3.1, equation (4.1) reduces to

$$E[N] = \lim_{z \to 1} \frac{\partial}{\partial z} \frac{P_0(z)}{\psi} \begin{bmatrix} \psi - \lambda_0 \lambda_2 \lambda_3 (\widetilde{S}_1(\lambda_1 - \lambda_1 C(z) - 1) \\ + \lambda_1 \lambda_3 \overline{p} \sigma \widetilde{S}_1(\lambda_1 - \lambda_1 C(z) \\ \times (\widetilde{S}_2(\lambda_2 - \lambda_2 C(z) - 1) \\ + \lambda_1 \lambda_2 \widetilde{S}_1(\lambda_1 - \lambda_1 C(z) \\ \times (\widetilde{S}_3(\lambda_3 - \lambda_3 C(z) - 1) \\ \times (\overline{p} \sigma q \widetilde{S}_2(\lambda_2 - \lambda_2 C(z)) + \overline{\sigma}) \end{bmatrix} \tag{4.2}$$

Taking limit by L Hospital's rule and after a little simplification, it is fairly easy to obtain $E[N]$.

Theorem 4.2: The mean waiting time in the orbit for the bulk arrival retrial queueing $M^X/(G_1, G_2)/1$ model can be represented by following expression

$$E(W) = \frac{1}{\lambda E[X]} \left[\frac{\lambda_0 (\rho + E[X] - 1)(\psi' - \lambda_0 \rho_1)}{\nu (1 - \rho) \psi'} + \frac{\psi'' - \lambda_0 \left(\eta_1 + \dfrac{E[X^2] \rho_1}{E[X]} \right)}{2\psi'} \right]$$

where

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3.$$

Proof: To prove the theorem 4.2, we use the Little's formula as given

$$E[W] = \frac{E[N]}{\lambda E[X]} \tag{4.3}$$

where

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$$

We remark that $E[W]$ can be easily obtained from equation (4.3) in the light of theorem 4.1.

5. Special Cases of the $M^X/(G_1,G_2)/1$ Model

In this section, we deduce some special cases by assigning the appropriate values of varying parameters involved in the system to obtain performance measures which emphasizes to verify our investigated results.

Case I: $M^X/(G_1,G_2)/1$ Retrial Queue with Two Phases of Essential Service

In this case, we assign $\sigma = 1, p = 0$ and $\lambda_i = \lambda,\ 0 \le i \le 3.$

Then theorem 3.4 yields

$$P(z) = (1 - z) P_0(z) \times \frac{1}{\left[\begin{pmatrix} \widetilde{S}_1(\lambda - \lambda C(z)) \widetilde{S}_2(\lambda - \lambda C(z)) \\ \times (q \widetilde{S}_3(\lambda - \lambda C(z)) + \overline{q}) \end{pmatrix} - z \right]} \tag{5.1}$$

where $P_0(z)$ is given by following equation:

$$P_0(z) = P_0(1) \times \exp\left[-\frac{\lambda}{\nu}\int_z^1 \left(\frac{\begin{array}{c}1-[\tilde{S}_1(\lambda-\lambda C(u))\tilde{S}_2(\lambda-\lambda C(u) \\ (q\tilde{S}_3(\lambda-\lambda C(u))+\bar{q})]\frac{C(u)}{u}\end{array}}{\begin{array}{c}[\tilde{S}_1(\lambda-\lambda C(u))\tilde{S}_2(\lambda-\lambda C(u) \\ \times(q\tilde{S}_3(\lambda-\lambda C(u))+\bar{q})]-u\end{array}}\right)du\right] \tag{5.2}$$

It is interesting to remark here that the equation (5.1) coincides with the results obtained by Kumar and Arumuganathan [22] which shows that our contribution of present research is generalization of results investigated by Kumar and Arumuganathan [22]

Case II: M^X/G/1 with Batch Arrival Retrial Queue

In this case, we set $q, \sigma = 1,\ p = 0,\ \ \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) = 1,\ \tilde{S}_3(\lambda_3 - \lambda_3 C(z) = 1$ and $\lambda_i = \lambda,\ 0 \le i \le 3$.
In view of assigned values of varying parameters, theorem 3.4 reduces to

$$P(z) = \frac{(1-z)P_0(z)}{[\tilde{S}_1(\lambda - \lambda C(z)) - z]} \tag{5.3}$$

where $P_0(z)$ is given by following equation:

$$P_0(z) = P_0(1) \times \exp\left[-\frac{\lambda}{\nu}\int_z^1 \left(\frac{1-[\tilde{S}_1(\lambda-\lambda C(u))]\frac{C(u)}{u}}{[\tilde{S}_1(\lambda-\lambda C(u))-u]}\right)du\right] \tag{5.4}$$

we remark here that the results found in equations (5.3) and (5.4) coincide with the results obtained by Falin [16].

Case III: $M^X/(G_1,G_2)/1$ Retrial Queue with Two Phases of Essential Service and the Earlangian Vacation Time

Assigning values of $q,\ \sigma = 1,\ p = 0,\ \ \tilde{S}_2(\lambda_2 - \lambda_2 C(z)) = 1, \lambda_i = \lambda,\ 0 \le i \le 3$, we get

$$\tilde{S}_3(\lambda - \lambda C(z)) = \left(\frac{\mu k}{\mu k + \lambda(1 - C(z)}\right)^k. \tag{5.5}$$

Then theorem 3.1 yields

$$P(z) = \frac{(1-z)P_0(z)}{\left[\begin{array}{c}\tilde{S}_1(\lambda-\lambda C(z))\tilde{S}_2(\lambda-\lambda C(z) \\ \times\left\{q\left(\frac{\mu k}{\mu k+\lambda(1-C(z)}\right)^k+\bar{q}\right\}\end{array}\right]-z} \tag{5.6}$$

where $P_0(z)$ is given by following equation

$$P_0(z) = P_0(1) \times \exp\left[-\frac{\lambda}{\nu}\int_z^1 \left(\frac{1-\left[\begin{array}{l}\tilde{S}_1(\lambda-\lambda C(u))\tilde{S}_2(\lambda-\lambda C(u)\\ \times\left\{q\left(\frac{\mu k}{\mu k+\lambda(1-C(z)}\right)^k+\bar{q}\right\}\end{array}\right]\frac{C(u)}{u}}{\left[\begin{array}{l}\tilde{S}_1(\lambda-\lambda C(u))\tilde{S}_2(\lambda-\lambda C(u)\\ \times(q\left(\frac{\mu k}{\mu k+\lambda(1-C(z)}\right)^k+\bar{q})\end{array}\right]-u} \right) du \right] \quad (5.7)$$

6. Numerical Illustration for Special Cases of the $M^X/(G_1,G_2)/1$ Model

In this section, we present some numerical results using Mat lab in order to illustrate the effect of various parameters on the key performance of our model. For the effect of parameters $\lambda_i; i=0,1,2,3; p,q,\sigma,\nu$ on the system performance measures, two dimensional graphs are drawn in Fig 1-8. We assume that the service time distributions for chosen parametric values are Erlangian, deterministic and Gamma distribution. Varying trends in mean orbit size for $M^X/E_k/1$, $M^X/\gamma/1$ and $M^X/D/1$ models have been examined.

For different types of models, first and second moments of service times are given as follows:

I. For $M^X/E_K/1$ Model

$$\left[E[S_1]=\frac{1}{\mu}\right] \; and \; \left[E[S_i^1]=\frac{(k+1)}{k\mu^2}\right], \; i=1,2,3.$$

II. For $M^X/D/1$ Model

$$\left[E[S_i]=\frac{1}{\mu}\right] \; and \; \left[E[S_i^1]=\frac{1}{\mu^2}\right], \; i=1,2,3.$$

III. For $M^X/\gamma/1$ Model

$$\left[E[S_i]=\frac{1}{\mu}\right] \; and \; \left[E[S_i^1]=\frac{k(k+1)}{k\mu^2}\right], \; i=1,2,3.$$

Figures 1-8 display the result for mean orbit size E[N] by varying the values of parameters $\lambda_i; i=0,1,2,3; p,q,\sigma,\nu$ for the different sets of Erlangian, Deterministic and Gamma service time distributions, respectively. In these figures, we observe that E[N] increases with the increasing values of λ_0, λ_1, λ_2, λ_3 and this increment is higher for Gamma distribution in comparison to Erlangian and Deterministic distributions.

For the different sets of service time distributions, the mean orbit size E[N] is compared with varying values of q,σ, p *and* ν in figures 5-8. It is clear from figures 5-6 that E[N] exhibits an increasing trend as we increase the values of q,σ. We observe that E[N] increases sharply in case of Gamma distribution. Figs 7-8 depict the decreasing pattern of mean orbit size for the increasing values of p and ν; the effect in case of Gamma distribution is more prominent

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than Erlangian and Deterministic distributions. Based on numerical experiment performed, overall we conclude that the varying trends in mean orbit size are more perceptible for Gamma service time distribution as compared to Erlangian and Deterministic service time distributions.

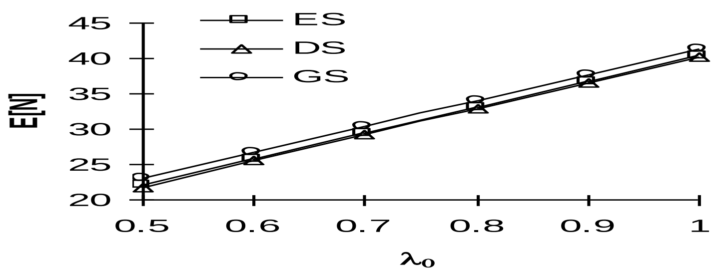

Fig. 1: E[N] versus λ_0

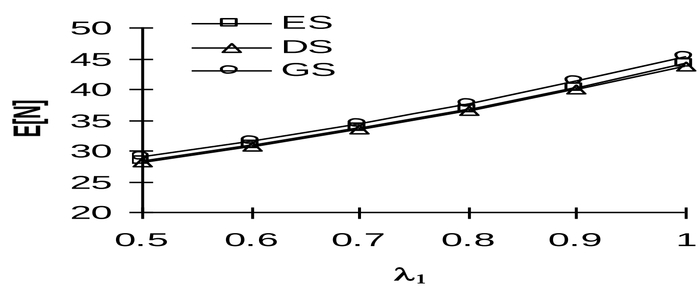

Fig. 2: E[N] versus λ_1

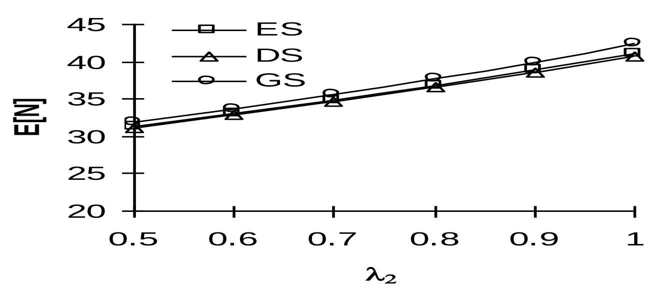

Fig. 3: E[N] versus λ_2

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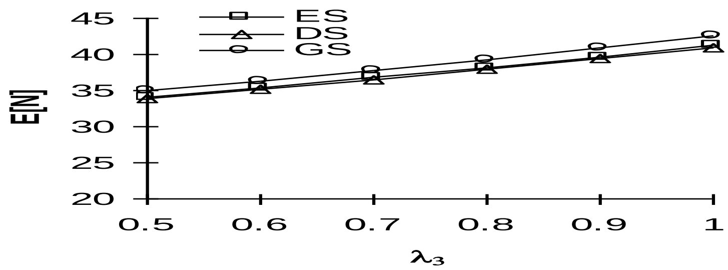

Fig. 4: E[N] versus λ_3

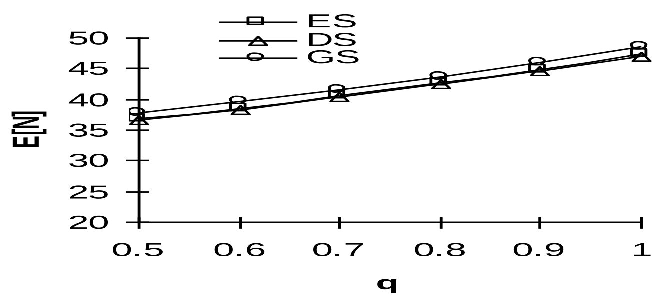

Fig. 5: E[N] versus q

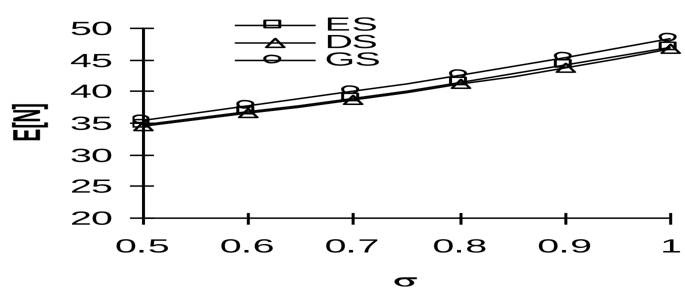

Fig. 6: E[N] versus σ

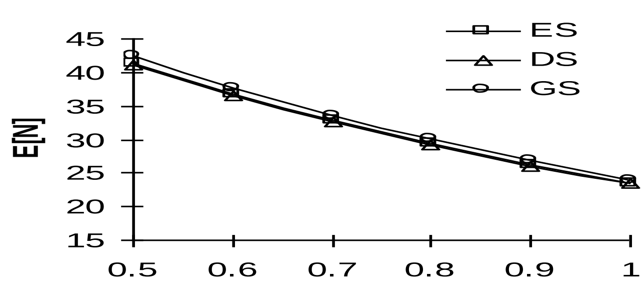

Fig. 7: E[N] versus p

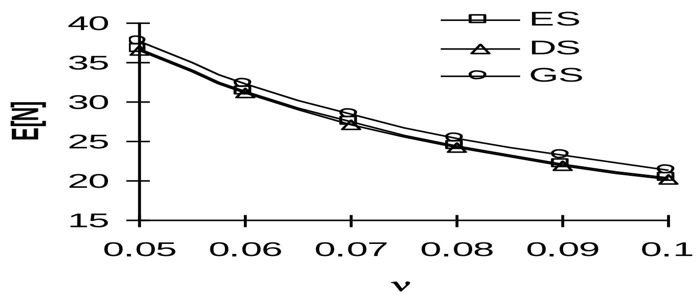

Fig. 8: E[N] versus ν

7. Conclusions

In the present paper, some significant performance measures of bulk arrival retrial queue with essential and optional services under Bernoulli vacation schedule have been presented successfully by way of using theorem 3.1. Specifically, various performance measures such as system length, waiting time and special cases are analyzed in a novel way. Some numerical illustrations and sensitivity analysis for three different types of service time distributions such as Deterministic, Exponential and Gamma distributions have been discussed successfully and it is keenly observed that our results for special cases discussed in this paper agree with results obtained by Falin [16] and Kumar and Arumuganathan [22]. In our sensitivity analysis, the effects of various parameters on the performance measures are illustrated numerically as well as graphically. We remark here that the results of the present paper are useful for the researchers, network design engineers and software engineers to design various computer communication systems. Moreover, it is remarkable here that the research of the present investigation can be extended more by introducing the concept of maximum entropy principle (MEP), server breakdown and multi-optional services. In this continuation, we note that the maximum entropy principle has been used recently by Maurya [27] to examine the steady state behaviour of bulk arrival retrial queueing $M^X/(G_1,G_2)/1$ model with second phase optional service and Bernoulli vacation schedule. Finally, with passing remarks we emphasize here that the present distinct contribution to scientific knowledge comprises fresh interpretation of facts as well generalization of research contribution of Falin [16] and Kumar and Arumuganathan [22] with an extended version of Maurya [26].

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Acknowledgement

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